Frames and Gabor Wavelets

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- A simple technical point:
  - With sufficient sampling density, a discrete family of Gabor wavelets is a frame, so the representation is 1-1
  - Gabor frames can be made to be (simultaneously) very snug and highly redundant
  - In cortex, redundancy is crucial for representational precision in the face of low SNR. Snugness comes for free
  - On the computer, redundancy may be undesirable, and 1-1 can be achieved by means other than redundancy (e.g., orthogonal wavelets, which are tight)

Discrete Gabor wavelets:

$$s(x) \in L^2(\mathcal{R}^2) \rightarrow S_{fp} = \langle g_{fp}, s \rangle \in \ell^2(\mathcal{R}^4)$$

where

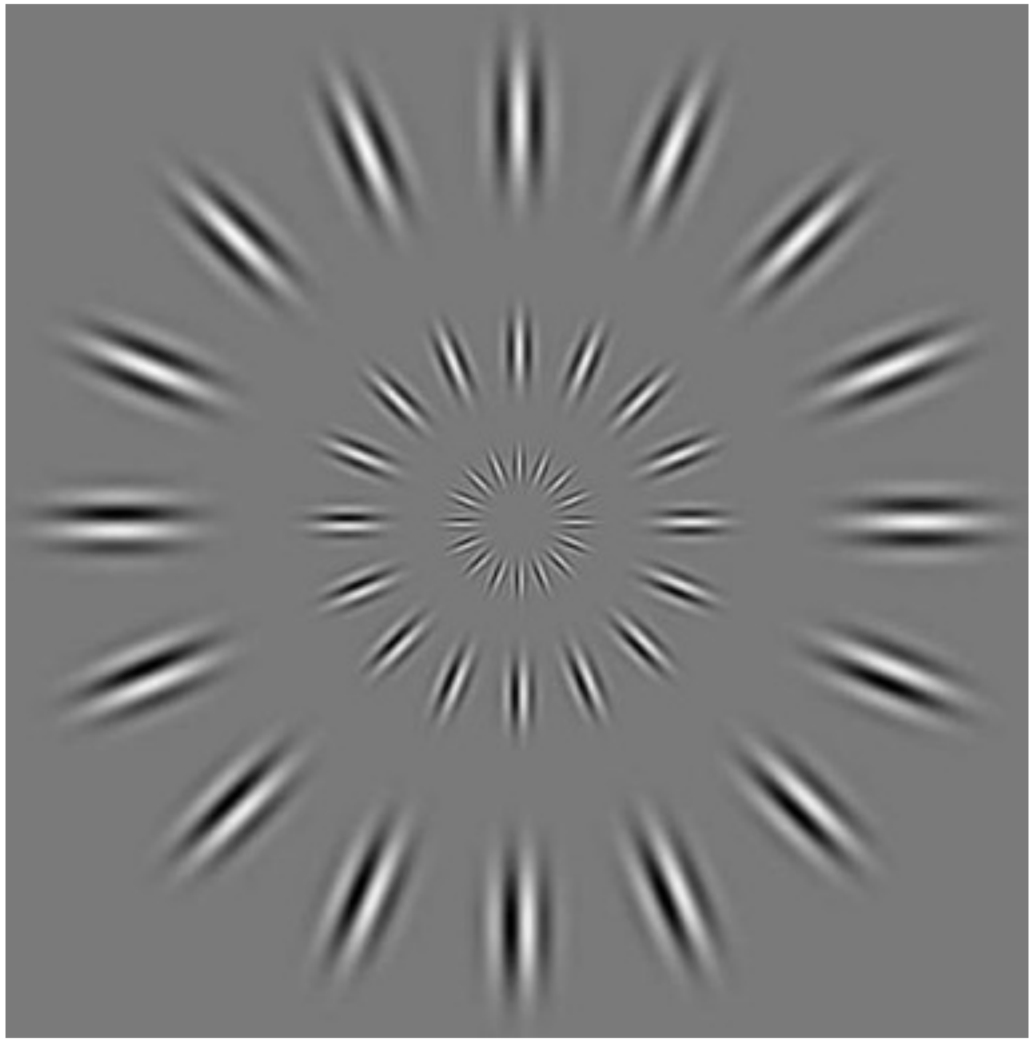
$$g_{fp}(x) \propto e^{-\frac{1}{2}\left(\frac{\omega}{\kappa}\right)^2 x^T \Sigma^{-1} x} \left(e^{i\omega u^T(\theta) p} - \ e^{-\frac{\kappa^2}{2}}\right)$$

$f = (\omega, \theta)$ is a point on the frequency plane

$p = (a, b)$ is a point on the image plane, and

$$\Sigma^{-1} = I - u(\theta) u^T(\theta)$$

$$u(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$



A wavelet $g_{sp}(x)$ is a *frame* if there exist $A > 0$ and $B < \infty$ such that

$$A\|s\|^2 \leq \sum_{f,p} |S_{fp}|^2 \leq B\|s\|^2$$

For any frame $g_{sp}(x)$, there exists a *dual frame* $h_{sp}(x)$ such that

$$s(x) = \sum_{f,p} S_{fp} h_{sp}(x)$$

Frame property implies 1-1 representation: discriminative, complete

B/A measures the conditioning of the problem of computing $\{h_{sp}(x)\}$ from $\{g_{sp}(x)\}$.
S*nug* frame:

$$\frac{B}{A} \approx 1 \quad \Rightarrow \quad h_{s,p}(x) \approx \frac{2}{A+B}\, g_{s,p}(x)$$

Well-conditioned and easy!

With unit-energy wavelets, $(A+B)/2$ measures frame redundancy. **With large redundancy, each S_{fp} can be encoded with lower precision** for similar reconstruction quality and representational accuracy

A Gabor frame is tightened (made more snug) by increasing both A and B and making them closer to each other: For Gabor, **snugness comes with redundancy**

- *Cortex*:
  - high redundancy is needed because of low representational precision
  - snugness comes for free
- *Computer*:
  - low redundancy is OK because of high representational precision
  - A loose frame requires more work for reconstruction
  - conditioning is less critical
  - choose between conciseness (orthogonal) and other desiderata (space/frequency localization, …)
- On either architecture, reconstructability (1-1) is useful, but we may not need reconstruction